Eur. Phys. J. B **23**, 437–440 (2001)

THE EUROPEAN PHYSICAL JOURNAL B

EDP Sciences

Energy gaps for fractional quantum Hall states described by a Chern-Simons composite fermion wavefunction

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Received 9 March 2001

Abstract. The Jain's composite fermion wavefunction has proven quite succesful to describe most of the fractional quantum Hall states. Its mathematical foundation lies in the Chern-Simons field theory for the electrons in the lowest Landau level, despite the fact that such wavefunction is different from a typical mean-field level Chern-Simons wavefunction. It is known that the energy excitation gaps for fractional Hall states described by Jain's composite fermion wavefunction cannot be calculated analytically. We note that analytic results for the energy excitation gaps of fractional Hall states described by a fermion Chern-Simons wavefunction are readily obtained by using a technique originating from nuclear matter studies. By adopting this technique to the fractional quantum Hall effect we obtained analytical results for the excitation energy gaps of all fractional Hall states described by a Chern-Simons wavefunction.

PACS. 73.43.-f Quantum Hall effects – 73.21.-b Electron states and collective excitations in multilayers, quantum wells, mesoscopic, and nanoscale systems

The discovery of the fractional quantum Hall effect (FQHE) [1,2] has stimulated extensive studies on the two-dimensional (2D) quantum many-electron system subject to a strong perpendicular magnetic field. Considerable progress has been made in understanding that the FQHE is essentially a strongly correlated incompressible liquid state [3], which exhibits an energy gap at some special densities of the 2D electronic system. The dominant sequence of FQHE states occurs when filling factor of the lowest Landau level (LLL) is $\nu = p/(2m\ p+1)$, where $p = 1, 2, \ldots$ and $m = 1, 2, \ldots$ are integers.

The model of choice consists of N interacting fully spin-polarized (spinless) electrons embedded in a uniform positive neutralizing background. The electrons with charge $-e$ ($e > 0$) and mass $m_{\rm e}$ move in a 2D plane and are subjected to a perpendicular uniform magnetic field. In the symmetric gauge the magnetic field $\mathbf{B} = (0, 0, B)$ is generated by the vector potential $\mathbf{A}(\mathbf{r}) = (-By/2, Bx/2, 0)$. The quantum mechanical Hamiltonian of this system can be written as $\hat{H} = \hat{K} + \hat{V}$, where $\hat{K}$ is the kinetic energy operator

$$\hat{K} = \frac{1}{2m_{\rm e}} \sum_{j=1}^{N} \left[-{\rm i}\hbar \boldsymbol{\nabla}_j + e\mathbf{A}(\mathbf{r}_j)\right]^2 \ , \qquad (1)$$

and

$$\hat{V} = \sum_{j<k}^{N} v(|\mathbf{r}_j - \mathbf{r}_k|) - \rho(\nu) \sum_{j=1}^{N} \int {\rm d}^2 r\ v(|\mathbf{r}_j - \mathbf{r}|) + \frac{\rho(\nu)^2}{2} \int {\rm d}^2 r_1 \int {\rm d}^2 r_2\ v(|\mathbf{r}_1 - \mathbf{r}_2|) \ , \qquad (2)$$

is the total electron-electron, electron-background and background-background interaction potential. The interaction potential is of Coulomb form, $v(|\mathbf{r}_j - \mathbf{r}_k|) = \left(e^2/4\pi\epsilon_0\epsilon|z_j - z_k|\right)$ where $z_j = x_j + {\rm i}y_j$ is the location of the jth electron in complex coordinates, $\rho(\nu)$ is the electronic density and ϵ is the dielectric constant of the background.

Many essential features of the FQHE are understood straightforwardly in the composite fermion (CF) picture due to Jain. [4] Intuitively one imagines the CF as being a bound state of an electron with an even number of vortices of the many-body quantum wavefunction [5] formed at the electronic densities $\rho(\nu) = \nu/[2\pi l_0(B)^2]$ where $l_0(B) = \sqrt{\hbar/(eB)}$ is the electronic magnetic length. The fundamental property of the CF-s is that they experience a reduced effective field, $B^* = B(1 - 2m\nu)$ where the effective filling factor of CF-s is an integer number $p = 1, 2, \ldots$ that corresponds to stable electronic filling factors $\nu = p/(2m\ p+1)$.

The theoretical approach that has proved most useful to understand the formation of CF-s is the fermion

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Chern-Simons (CS) field theory [6], itself an mathematical outgrowth of the more intuitive picture developed by Jain. In the fermion CS approach, the electron system is subjected to a mathematical transformation which converts it into a new system of fermions coupled to a new fictitious CS vector potential. Such transformation corresponds to the formation of the CF-s in Jain's picture. Although both schemes are based on the same physics, the Jain's CF wavefunction is different from the CS wavefunction that at a mean field level has the form

$$\Psi_\nu = \prod_{j<k}^{N} \frac{(z_j - z_k)^{2m}}{|z_j - z_k|^{2m}} \Phi_p(B^*) \ , \tag{3}$$

where $\Phi_p(B^*)$ is the Slater determinant wavefunction of p filled CF Landau levels evaluated at the magnetic field shown in the argument. The main concept behind the CS wavefunction is the attachment of flux tubes to electrons. Such flux tubes are described by the Jastrow factor that in this case is a pure phase factor. The CS wavefunction is not the best representation for the actual electronic state, since it has a large mixing with higher LL-s and no special short-distance correlations. The correct physics of the FQHE is attained by binding of vortices only. Such state can be obtained by throwing away the denominator of equation (3) and then performing a projection of the whole wavefunction into the LLL. The attachment of vortices, instead of flux tubes is the fundamental element in Jain's CF theory. Such theory has been able to provide accurate wavefunctions for the ground state and excited states that give good estimates for the energy gaps that are one of most important measurable quantities from the experimental point of view. It is well known that the calculation of the energy excitation gap for Jain's composite fermion wavefunction cannot be carried out analytically and, in general, the calculation of energy gaps arising from such many-body phenomena is a complex process full of technical difficulties.

Unlike the Jain's CF wavefunction, we note that energy gaps can be analytically calculated in the case of the fermion CS wavefunction by adopting a well known technique [7] previously used in the studies of nuclear matter. In view of the qualitative differences between the CS wavefunction and Jain's wavefunction we point out that the CS wavefunction is not a very realistic description of the FQHE. However the CS wavefunction is still of interest since it provides a many-body model, where the excitation gap properties can be calculated analytically in closed form. The mathematics is not elaborate, but the results are quite instructive and will allow us to provide analytical results for the excitation gap energy, $E_g(\nu)$ of the FQHE states at filling factor $\nu = p/(2m\ p+1)$ described by the CS wavefunction. The excitation gap energy is defined as the energy to add one quasiparticle and one quasihole, far away to each other, to the quantized Hall groundstate at the given filling factor. In our calculations, we consider only the unprojected energy gap that comes from the Coulomb correlations, without attempting to estimate any kinetic energy effect due to the Landau level mixing that is inherent in the CS wavefunction. For filling ν this corresponds to the quasiparticle-quasihole excitation obtained by promoting a single CF from the uppermost filled CF Landau level to the lowest empty CF Landau level.

For filling factor $\nu = p/(2m\ p+1)$ we have p CF Landau levels filled, where the uppermost filled Landau level has quantum index $(p-1)$. Promoting a CF from the Landau level with quantum index $(p-1)$ to the one with quantum index p will produce a correlated wavefunction that describes the quasiparticle-quasihole excitation. Following the formalism of Friedman and Pandharipande [7] it is more convenient to calculate the quasiparticle-quasihole excitation energies by thinking as removing not one single CF, but a small number $\Delta N = xN (\ll N_s^*)$ of CF-s from the uppermost filled to the lowest empty CF Landau level and defining the quasiparticle-quasihole excitation energy as

$$E_g(\nu) = \lim_{\Delta N \to 0} \frac{\partial}{\partial \Delta N} \Delta U(x,N) = \lim_{x\to 0} \frac{\partial}{\partial x} \left[\frac{\Delta U(x,N)}{N} \right] \ , \tag{4}$$

where $\Delta U(x,N)$ is the change of the total correlation energy of the system and x is the fraction of displaced CF-s. The change of the correlation energy per particle is then given by

$$\frac{\Delta U(x,N)}{N} = \frac{1}{N} \frac{\langle \Psi_\nu^{\rm ph}(x)|\hat{V}|\Psi_\nu^{\rm ph}(x)\rangle}{\langle \Psi_\nu^{\rm ph}(x)|\Psi_\nu^{\rm ph}(x)\rangle} - \frac{1}{N} \frac{\langle \Psi_\nu|\hat{V}|\Psi_\nu\rangle}{\langle \Psi_\nu|\Psi_\nu\rangle} \ , \tag{5}$$

where $\Psi_\nu^{\rm ph}(x)$ is the wavefunction that describes the quasiparticle-quasihole pair. One writes the quasiparticle-quasihole excitation energy as

$$E_g(\nu) = \lim_{x\to 0} \frac{\partial}{\partial x} u_\nu(x) \ , \tag{6}$$

where $u_\nu(x)$ is the correlation energy per particle corresponding to the quasiparticle-quasihole wavefunction and is given by

$$\begin{aligned} u_\nu(x) &= \frac{1}{N} \frac{\langle \Psi_\nu^{\rm ph}(x)|\hat{V}|\Psi_\nu^{\rm ph}(x)\rangle}{\langle \Psi_\nu^{\rm ph}(x)|\Psi_\nu^{\rm ph}(x)\rangle} \\ &= \frac{\rho(\nu)}{2} \int {\rm d}^2 r_{12} \left[g_\nu(x, r_{12}) - 1 \right] v(r_{12}) \ , \end{aligned} \tag{7}$$

where the radial distribution function $g_\nu(x, r_{12})$ is defined as

$$g_\nu(x, r_{12}) = \frac{N(N-1)}{\rho(\nu)^2} \frac{\int {\rm d}^2 r_3 \cdots {\rm d}^2 r_N \ |\Psi_\nu^{\rm ph}(x)|^2}{\int {\rm d}^2 r_1 \cdots {\rm d}^2 r_N \ |\Psi_\nu^{\rm ph}(x)|^2} \ , \tag{8}$$

and for the system under consideration will depend only on the 2D interparticle spatial distance $r_{12} = |\mathbf{r}_1 - \mathbf{r}_2|$ and variable x. In the present case one observes that $|\Psi_\nu^{\rm ph}(x)|^2 = |\Phi_p(x, B^*)|^2$, and this corresponds to a squared Slater determinant that represents $(p-1)$ totally filled CF Landau levels, the pth CF Landau level (with

CF Landau level index $(p-1)$) not totally filled having been removed a small fraction of $\Delta N = xN(\ll N_s^*)$ CF-s and the $(p+1)$th CF Landau level (with CF Landau level index p) that is partially filled with the small fraction x of CF-s that were removed from the underlying level. It is well known that for fully spin-polarized electrons (CF-s)

$$g_\nu(x, r_{12}) = 1 - |l_p(x, r_{12})|^2 \ , \qquad (9)$$

where the "statistical exchange" factor is computed from $l_p(x, r_{12}) = \hat{\rho}_p(x, \mathbf{r}_1, \mathbf{r}_2)/\rho(\nu)$. The electronic density $\rho(\nu)$ is the same as the CF's density and the (reduced) one-body density matrix $\hat{\rho}_p(x, \mathbf{r}_1, \mathbf{r}_2)$ which corresponds to the dynamically uncorrelated state $\Phi_p(x, B^*)$ is given by

$$\hat{\rho}_p(x, z_1, z_2) = \sum_{n=0}^{p-1} \sum_{l=0}^{N_s^*-1} \varphi^*_{n,l}(z_1)\varphi_{n,l}(z_2) + \frac{\Delta N}{N_s^*} \left[\sum_{l=0}^{N_s^*-1} \varphi^*_{p,l}(z_1)\varphi_{p,l}(z_2) - \sum_{l=0}^{N_s^*-1} \varphi^*_{(p-1),l}(z_1) \cdot \varphi_{(p-1),l}(z_2) \right] , \qquad (10)$$

where $\Delta N/N_s^* = x\ p$ is the average occupancy of a quantum state in the partially filled CF Landau levels. In the above equation, the sum is extended over all CF states, where $\varphi_{n,l}(z)$ are the single particle states of the ideal 2D Hamiltonian, where $n = 0, 1, \ldots, (p-1)$ denotes the various CF Landau levels at magnetic field B^* and $l = 0, 1, \ldots, (N_s^* - 1)$ is the angular momentum quantum number for the CF-s. We then follow the same mathematical procedure as previously used [8] to calculate the groundstate energy of fractional Hall states described by the groundstate CS wavefunction. We skip the lengthy algebra and just report the final analytic formula for the correlation energy gap of states with filling factor $\nu = p/(2mp+1)$ described by the CS wavefunction,

$$E_g(\nu) = -\frac{1}{\sqrt{2mp+1}} \int_0^\infty \mathrm{d}t \exp(-\frac{t^2}{2}) L^1_{p-1}(\frac{t^2}{2}) \times \left[L_p(\frac{t^2}{2}) - L_{p-1}(\frac{t^2}{2}) \right] \frac{1}{4\pi\epsilon_0} \frac{e^2}{\epsilon l_0(B)} \ , \qquad (11)$$

where $L_n(x) = \frac{\mathrm{e}^x}{n!} \frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^n \mathrm{e}^{-x})$ are the Laguerre polynomials of order $n = 0, 1, \ldots$ and $L_n^k(x) = (-1)^k \frac{\mathrm{d}^k}{\mathrm{d}x^k}[L_{n+k}(x)]$ are the generalized Laguerre polynomials of order $n = 0, 1, \ldots$ and degree $k = 0, 1, \ldots$ We calculated exactly the integrals appearing in equation (11) for increasing values of p and $m = 1$ and $m = 2$.

In Figure 1 we show the correlation energy gaps for the for the sequence of fractional Hall states $\nu = p/(2p+1)$ (filled circle) and $\nu = p/(4p+1)$ (opaque circle) described by the CS wavefunction. We compare our analytical results for the CS wavefunction with more realistic calculations that use a fully LLL projected Jain's wavefunction [9]. As expected there is not much quantitative similarity between the two since one describes flux attachment and the other describes attachment of vortices, phenomena that are qualitatively different from each other. In the $p \to \infty$ limit the energy gaps vanish as it should happen for a fractional filling factor with even denominator.

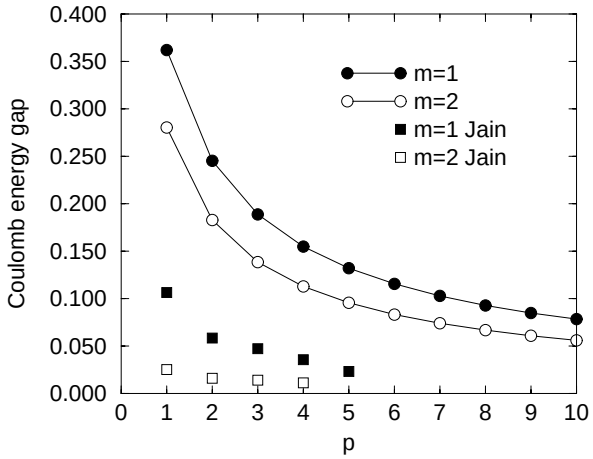

Fig. 1. The correlation energy gaps for FQHE states with filling $\nu = p/(2p+1)$ (filled circle) and $\nu = p/(4p+1)$ (opaque circle) described by the Chern-Simons (CS) wavefunction are compared with corresponding results, (filled square and opaque square) for the LLL projected Jain's wavefunction [9]. The energy gaps vanish to zero in the $p \to \infty$ limit that corresponds to even-denominator-filled states. The energies are expressed in the standard units of $\frac{1}{4\pi\epsilon_0} \frac{e^2}{\epsilon l_0(B)}$.

In Table 1 and Table 2 we show the values of the first ten energy gaps for states $\nu = p/(2p+1)$ and $\nu = p/(4p+1)$ respectevily, described by the CS wavefunction. Such analytical values are compared with corresponding Monte Carlo results for the LLL projected Jain's wavefunction.

We approximate the CS analytical results to numerical values with 6th decimal accuracy in order to provide accurate tabulated values for future comparisons. The provided tables can be used to gauge the accuracy of techniques and numerical schemes currently applied in the study of excited state properties of the FQHE.

In conclusion, we considered the FQHE states described by a CS many-body wavefunction. We found that this is a rare example of a many-body problem where we can exactly calculate both ground state [8] and excited state properties. We provide qualitative information as well as accurate quantitative estimates for this class of wavefunctions. Unfortunately, the CS wavefunction is not the most suitable wavefunction for the actual electronic states of the system. A better wavefunction to study the FQHE is Jain's wavefunction that is obtained by arbitrarily throwing away the denominator of equation (3) and when necessary projecting into the LLL. Any of these operations complicates the structure of the wavefunction to the extent that exact analytical results for groundstate and/or excited state properties cannot be obtained any longer.

Table 1. We compare our analytic results for the excitation energy gap corresponding to the Chern-Simons (CS) wavefunction with corresponding results for the LLL projected Jain's wavefunction [9] at filling $\nu = p/(2p+1)$. The energy gap is expressed in units of $\frac{1}{4\pi\epsilon_0}\frac{e^2}{\epsilon l_0(B)}$, where $l_0(B)$ is the magnetic length of the electrons.

m	p	$\nu = \frac{p}{2mp+1}$	CS wavefunction	Jain's wavefunction
1	1	1/3	$\frac{1}{2}\sqrt{\frac{\pi}{6}}$=0.361801	0.1063
1	2	2/5	$\frac{7}{16}\sqrt{\frac{\pi}{10}}$=0.245218	0.0585
1	3	3/7	$\frac{51}{128}\sqrt{\frac{\pi}{14}}$=0.188743	0.0474
1	4	4/9	$\frac{759}{2048}\sqrt{\frac{\pi}{18}}$=0.154828	0.0356
1	5	5/11	$\frac{11445}{32768}\sqrt{\frac{\pi}{22}}$=0.131986	0.0231
1	6	6/13	$\frac{87069}{262144}\sqrt{\frac{\pi}{26}}$=0.115455	...
1	7	7/15	$\frac{666743}{2097152}\sqrt{\frac{\pi}{30}}$=0.102883	...
1	8	8/17	$\frac{20525111}{67108864}\sqrt{\frac{\pi}{34}}$=0.0929696	...
1	9	9/19	$\frac{5709117897}{19327352832}\sqrt{\frac{\pi}{38}}$= 0.0849336	...
1	10	10/21	$\frac{24584746575}{85899345920}\sqrt{\frac{\pi}{42}}$= 0.0782756	...
...	...	...	...	...
1	∞	1/2	0	...

Table 2. We compare our analytic results for the excitation energy gap corresponding to the Chern-Simons (CS) wavefunction with corresponding results for the LLL projected Jain's wavefunction [9] at filling $\nu = p/(4p+1)$. The energy gap is expressed in units of $\frac{1}{4\pi\epsilon_0}\frac{e^2}{\epsilon l_0(B)}$, where $l_0(B)$ is the magnetic length of the electrons.

m	p	$\nu = \frac{p}{2mp+1}$	CS wavefunction	Jain's wavefunction
2	1	1/5	$\frac{1}{2}\sqrt{\frac{\pi}{10}}$= 0.28025	0.0253
2	2	2/9	$\frac{7}{16}\sqrt{\frac{\pi}{18}}$= 0.182775	0.0162
2	3	3/13	$\frac{51}{128}\sqrt{\frac{\pi}{26}}$= 0.1385	0.0142
2	4	4/17	$\frac{759}{2048}\sqrt{\frac{\pi}{34}}$= 0.112654	0.0113
2	5	5/21	$\frac{11445}{32768}\sqrt{\frac{\pi}{42}}$= 0.0955248	...
2	6	6/25	$\frac{261207}{786432}\sqrt{\frac{\pi}{50}}$= 0.0832556	...
2	7	7/29	$\frac{66743}{2097152}\sqrt{\frac{\pi}{58}}$= 0.0739928	...
2	8	8/33	$\frac{20525111}{67108864}\sqrt{\frac{\pi}{66}}$= 0.0667281	...
2	9	9/37	$\frac{5709117897}{19327352832}\sqrt{\frac{\pi}{74}}$= 0.0608633	...
2	10	10/41	$\frac{24584746575}{85899345920}\sqrt{\frac{\pi}{82}}$= 0.0560201	...
...	...	...	...	...
2	∞	1/4	0	...

This work was supported by the University of Missouri Research Board and Research Council.

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